Symmetry, Physical Theories and Theory Changes V1

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Abstract

We discuss the problem of theory change in physics. Particularly, we consider the theory of kinematics of particles in distinct space-time backgrounds. We propose a characterization of the concept of physics theory based on symmetries of these kinematics. The concepts and tools of the theory of groups will be extensively used. The proposed characterization is compatible with the modern ideas in philosophy of science – e.g. the semantic approach to a scientific theory. The advantage of our approach lies in it being conceptually simple, allowing an analysis of the problem of the mathematical structure and hints at a logic of discovery. The problem of theory change can be framed in terms of the notions of Inönü-Wigner contraction/extension of groups of symmetry. Furthermore, from the group of symmetry the kinematic equations or physics laws associated with the underlying physical theory can be obtained – this is notoriously known as the Bargmann-Wigner program.

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I. INTRODUCTION

A major task in the philosophy of science is to provide a characterization of a scientific theory. In the 20th century there were striking developments in this field. The conceptual by-products developed then are nowadays well-known. From the 30's to the 50's, appeared the *syntactic* approach; next in the 60's, the *semantic* approach, followed finally, in the 70's, by the *structuralist* approach. All of these approaches share a general perspective in the search of a meta-theory of the scientific theory. Not surprisingly, they are heavily based on the logic resources[1]. Nowadays the semantic approaches are certainly dominant. As examples, we may cite the works of Bas van Frassen [29] and Steve French and Newton da Costa [10]. Nevertheless, a growing number of works have emphasized some of their limitations [2].

This work presents a different perspective for the characterization of a scientific theory. We concentrate on physical theories which offer the advantage of a great precision of their characterizations: the availability of their underlying mathematical structures allows good determination and description of their essential aspects. We will be even more focused and consider a particular class of physical theories based on the existence of *symmetry groups*. A main idea of this work is to characterize a physical theory by a pair of labels consisting of a group of symmetry and a domain, characterized by a certain type of representation of the group.

Inspired by van Frassen's work [30], we reassess in this new perspective the problem of theory change. Moreover, this reassessment hints at a " logic of discovery " [20]. In fact, the new foundations lied by characterizing a physical theory in terms of groups of symmetry present the constitutive elements of a theory change: in this context, theory changes are well encapsulated by the notion of Inönü-Wigner contraction/extension. Here we present explicit examples guided by symmetries, to demonstrate their heuristic value, and the pertinence of our proposed characterization.

We emphasize that this approach captures the way physicists construct and present their works. It is based on the successful program settled by Wigner, Inönü and Bargmann, later completed by Bacry and Lévy-Leblond, for the theory of free particles. Starting from a kinematics covariant (INVARIANT ? ***) Êunder Galilean group, we end up with a kinematics covariant (INVARIANT ? ***) under (anti-)deSitter group. This progression, that we present here, may be seen as a path towards *simplicity*: this concept of group theory (defined below) is also dubbed "more fundamental" by physicists.

Section 2 *** delimits *** the boundaries of our discussion to those physical theories that

[1] Recent studies of a particular foundation of physical theories [27] are still developed in an exclusively logic approach.

[2] Cf. [16]. Frigg, as quoted by Halvorson [16], had already pointed out in a seminal paper [13] the crucial limitations regarding the possibility of the semantic conceptions to be able to account for the whole complexity involved in the structure of the scientific theories.

are suitable to be described in terms of symmetry groups. We introduce their characterization in terms of a group of kinematical symmetries with its possible representations.

Section 3 provides the necessary background of the notions of Inönü-Wigner contraction/extension at the basis of our conception of theory change. Once these basis are settled, we engage in a tour from Galilean groups to (anti-)deSitter groups, via Poincaré group, in order to illustrate the theory change for the kinematics of a free particle. Section 4 closes this work with a discussion and concluding remarks.

An appendix briefly presents the main aspects of the Bargmann-Wigner program.

II. PHYSICS, PHYSICAL THEORY AND DOMAINS

Focusing on the discipline of physics, we propose a minimal characterization of the nature of a physical theory, through a set of labels attached to it. Such a labeling process is not sufficient to entirely account for the intricate web of constructs, processes and interpretations of a physical theory, and cannot be seen as a general definition. We believe however that any general definition (of a physical theory) must somehow encompass the proposed characterization. Leaving that for future work, we emphasize the pragmatical character of our approach to explicitly discuss the problem of theory change.

Following a position in a sense pioneered by A. Einstein, we characterize a physical theory guided by the physical principle (or postulate) that states: "*the laws of physics are the same (does not change) for any* **class of inertial observers** [3]". The remarkable fact is that a **class of inertial observers** appears as a symmetry class of the kinematic group, i.e., the group of transformations mapping one representative to another, with composition law [4].

We now set the boundaries for the present discussion. We focus on the physical theories of non-interacting (equivalently, free) particles, whose motions are geodesics of some geometric background (we do not consider fields, strings and/or similar/derived objects). In this scenario, we characterize a physical theory by a pair of labels consisting of a **domain** and a kinematic **symmetry**:

[3] Observer here is equivalent to a system of detectors or measuring apparatus.
[4] There are of course other possible algebraic structures one can also consider, e.g. [15].

Symmetries

The first label of a physical theory consists of its [kinematical] symmetry: Galilean (Newtonian), Poincaré (special relativity) and deSitter/anti-deSitter (cosmological).

Domains

The second label — to which we refer as the domain — refers to the type of representation one considers for the symmetries of the theory: the classical or the quantum regime.

A group may be defined in an abstract way as a set endowed with an operation law satisfying certain axioms [3, 14]. On the other hand, it may be *represented* by its linear action on a linear space, on which its elements act as operators; the operation law is identified with the composition. The representation is defined (fonctorially) as the group homomorphism from the original abstract group to the group of endomorphisms of the linear space. We can then handle the elements of a group as transformations acting on the underlying linear space associated to a physical system.

We restrict ourselves to two types of linear representations, defining two possible **domains of a theory**. In the *classical domain*, the group is considered as a kinematic symmetry. It acts as canonical transformations on the "phase space " endowed with a symplectic structure [11]. In the *quantum domain* it acts (unitarily) on a Hilbert space of states. This is also known as (anti-)unitary representation.

The *quantization* procedure is defined as the transition from the classical domain to the quantum one. It may be accomplished in different ways, e.g., Dirac quantization, geometric quantization, deformation quantization and so on. The reciprocal operation may be phrased as "taking the classical limit". We do not discuss these procedures here but give an insight of quantization in xxx.

In an idealized sense, a given symmetry together with a domain is equivalent to the ensemble of all the constructs of a theory, that is, the physical quantities like mass, spin, equations of motion and so on. Such constructs can be obtained from the representation theory of the physical theory, according to the strategy known as the Wigner-Bargmann program. See the appendix for a small discussion of this program.

III. CONTRACTIONS AND EXTENSIONS OF GROUPS

We apply the characterization of physical theories based on the concept of symmetry to the problem of theory change: the passage from a physical theory to another. This notion involves a **principle of correspondence**. Loosely speaking, this principle encompasses the idea that one can move forward from a certain point to another as long as the move back way is known: the idea of moving forward is intimately connected and dependent on the ability to move backward. In another guise, we are stating that it is the step back that allows for the jumping ahead.

For an intrepid navigator about to sail across unknown oceans, it would be a suicidal journey if he could not sail against the wind and sea streams. He should keep the ability to come back home. The example of the Kon-tiki and its journey across the South Pacific comes to mind here. Other examples are the Ariadne's thread or the Hans and Gretchen's bread crunches (those taught us that there might be other problems to face, like a bird eating the crunches).

The analog here of the unknown ocean or of the Minotaur's labyrinth may be seen as the landscape of physical theories: generalizing a theory is a kind of navigation in this landscape and one must keep the ability of restoring the initial theory from its generalization. This prescription may be related to two principles of the XXth century physics: the principle of correspondence evoked by Einstein in his work on special relativity, that we associate with the symmetry label characterizing a theory; and on the other hand the principle of correspondence stated by Bohr in the dawn of quantum theory which focusing — one could argue — on the domain label[5].

The following illustrates the application of the principle of correspondence, in combination with our characterization of a physical theory in terms of its kinematical group. Our main tools will be the Inönü-Wigner contraction and extension of Lie groups (described generally below). We study the Inönü-Wigner extension from a Newtonian group to a cosmological group as an instance of theory change.

[5] We thank A. Polito to call our attention to the parallel between the proposed characterization of a theory and the two statements of the principle of correspondence proposed by Einstein and Bohr.

A. Group contractions

The Inönü-Wigner contraction of a Lie group [18] is best described in terms of its associated Lie algebra which can be seen as the Lie group infinitesimal counterpart. The Inönü-Wigner contraction is a process allowing one to construct a new Lie algebra not isomorphic to the initial one, but preserving some of its structure. It proceeds by singular transformations of the infinitesimal elements (the generators) and, in this sense, it can be generalized to other algebraic structures [19]. Starting from a Lie algebra $\mathfrak{g}$, one constructs a parametrized family of new algebras, $\mathfrak{g}_\epsilon$, which are isomorphic to $\mathfrak{g}$ for $\epsilon \neq 0$, but not for the singular value $\epsilon = 0$.

The algebras $\mathfrak{g}_\epsilon$, for $\epsilon \neq 0$, are obtained by reparametrizations of $\mathfrak{g}$. Then, the new Lie algebra emerges from a singular limit of the parameter ϵ, that is, $\epsilon \to 0$. This new Lie algebra generates in turn a new Lie group via the exponential map (2). The process of contraction may be seen as a special case of *degeneration* [6].

A Lie algebra $\mathfrak{g}$ is conveniently described by a family of generators J_i, with $i = 1, ..., N$ (its dimension), together with commutation relations

$$[J_i, J_j] = f_{ijk}\; J_k, \tag{1}$$

where the antisymmetric operation $[\cdot, \cdot]$ is called the *commutator* and the f_{ijk} the *structure constants*. A Lie algebra element expands as $a^i\; J_i$ (implicit index summation). And each Lie group element $g \in G$ is obtained through the exponential map[6]

$$\exp:\; \mathbb{R}^N \to G, \qquad (\alpha_i) \mapsto g(\alpha_i) = e^{i\alpha^i J_i}, \tag{2}$$

where $\alpha_i \in \mathbb{R}$, with $i = 1, ..., N$.

Concretely, let us assume that the Lie algebra $\mathfrak{g}$ contains a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We call $\mathfrak{p}$ the complement of $\mathfrak{h}$ in $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where the symbol $\oplus$ stands for direct sum of vector spaces. The defining commutators of the Lie algebra can then be schematically decomposed as

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} + \mathfrak{p}. \tag{3}$$

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[6] The exponential map allows one to obtain only those elements in the Lie Group continuously connected to the identity. This is sufficient for the ensuing discussion.

The reparametrization is the replacement of each generator $J \in \mathfrak{p}$ by a generator $J' = \epsilon\ J$, with $\epsilon \neq 0$. In abbreviated notation, $\mathfrak{p}$ becomes $\mathfrak{p}' = \epsilon\ \mathfrak{p}$. The algebra remains the same, but the commutation relations take the reparametrized form

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}'] = [\mathfrak{h}, \epsilon\ \mathfrak{p}] \subset \epsilon\ \mathfrak{p} = \mathfrak{p}', \qquad [\mathfrak{p}', \mathfrak{p}'] = [\epsilon\ \mathfrak{p}, \epsilon\ \mathfrak{p}] \subset \epsilon^2\ (\mathfrak{h} + \mathfrak{p}). \tag{4}$$

The singular limit $\epsilon \to 0$ gives a well-defined but different Lie algebra obeying

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}'] \subset \mathfrak{p}', \qquad [\mathfrak{p}', \mathfrak{p}'] = 0, \tag{5}$$

at the singular limit. Observe that $\mathfrak{p}'$ is now an Abelian algebra.

The new Lie algebra obtained is a *semi-direct product* of Lie algebras denoted by $\mathfrak{g}' = \mathfrak{h} \ltimes \mathfrak{p}'$, where the symbol $\ltimes$ means semi-direct product. If the second relation in (5) were $[\mathfrak{h}, \mathfrak{p}'] = 0$, then the new Lie algebra would be a *direct product* $\mathfrak{g}' = \mathfrak{h} \times \mathfrak{p}'$.

1. *Example of Inönü-Wigner Contraction*

We now give a simple example of the Inönü-Wigner contraction. The group $SO(3)$ of rotations in three-dimensional Euclidean space admits the Lie algebra $\mathfrak{so}(3)$. It is generated by J_i, $i = 1, 2, 3$, with commutation relations

$$[J_i, J_j] = \sum_{k=1}^{3} \epsilon_{ijk}\ J_k. \tag{6}$$

Here ϵ_{ijk} is the Levi-Civita symbol: $\epsilon_{123} = 1$ with odd permutations of the indexes changing sign, even permutations keeping sign, and the other components vanishing. Explicitly,

$$[J_1, J_2] = J_3, \qquad [J_2, J_3] = J_1, \qquad [J_3, J_1] = J_2. \tag{7}$$

We now define $\mathfrak{h}$ as the subalgebra generated by only J_3, so that J_1, J_2 generate $\mathfrak{p}$. We recognize the same structure as in (3) with $[J_3, J_3] = 0$ being an example of the schematic relation $[\mathfrak{h}, \mathfrak{h}] = 0$. We rescale the elements of $\mathfrak{p}$ by Λ as

$$j_1 = \Lambda\ J_1, \qquad j_2 = \Lambda\ J_2, \qquad j_3 = J_3, \tag{8}$$

and keep J_3 fixed. For non zero values of Λ, the algebra remains the same, although with the new expression for the non-vanishing commutators

$$[j_1, j_2] = \Lambda^2\ j_3, \qquad [j_2, j_3] = j_1, \qquad [j_3, j_1] = j_2. \tag{9}$$

Performing the contraction by taking the limit $\Lambda \to 0$, we obtain a new Lie algebra with three generators obeying the relations

$$[j_1, j_2] = 0, \qquad [j_2, j_3] = j_1, \qquad [j_3, j_1] = j_2. \tag{10}$$

They characterize the Lie algebra $\mathfrak{e}(2) = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ of the *two-dimensional Euclidean group* E(2). Note that $SO(2)$, the (special) orthogonal group, is the group of rotations of the two-dimensional Euclidean space and it is natural to associate to it the group of translations of the same space, to complete the isometries. This corresponds to the natural *augmentation* $SO(2) \to ISO(2) = E(2)$.

It is instructive to observe what happens to some invariant quantities of the Lie group. The $\mathfrak{so}(3)$ algebra admits the invariant R defined through

$$J_1^2 + J_2^2 + J_3^2 = R^2 \ \mathbb{1}, . \tag{11}$$

After the rescaling in (8) , this equation becomes

$$j_1^2 + j_2^2 + \Lambda^2 j_3^2 = \Lambda^2 \ R^2 \ \mathbb{1}. \tag{12}$$

The contraction breaks this equation into two independent parts

$$j_1^2 + j_2^2 = 0, \qquad j_3^2 = R^2. \tag{13}$$

B. Extensions

A group that cannot be written as a (semi-)direct product of subgroups is called a *simple group*. As illustrated by the example above, the Inönü-Wigner contraction diminishes the simplicity of the group.

There is an inverse procedure, called the Inönü-Wigner extension, which achieves simplicity. It extends a Lie group, composed as (semi-)direct product of two (or more) other Lie groups, towards a simpler group: a group with less (semi-)direct products of groups. For instance, the extension associated with the previous contraction may be written as $ISO(2) \to SO(3)$, so that we may write the diagram

$$\begin{array}{ccc} SO(2) & & \\ \big\downarrow{\scriptstyle \text{augm}} & & \\ ISO(2) & \xrightarrow{\text{ext}} & SO(3) \end{array}$$

Note that this diagram may be thought of as expressing the transition from pre-Newtonian physics to Newtonian physics, in the sense that the latter corresponds to the introduction of isotropic Euclidean space [23]. We will see below that this diagram may be further extended.

IV. STUDY OF CASE OF THEORY CHANGE: FROM NEWTON TO COSMOLOGY

The definitions and discussions of this section follow in some extent from the work of Lévy-Leblond and collaborators [1, 2, 24]. We display a broad overview of their comprehensive beautiful works on the classification of the kinematic groups. The kinematic group of a theory is the group of isometries of the space-time of that theory: the set of transformations preserving the metric of the space-time with the composition of transformations.

Along the way we can observe rather concrete examples of theory change, that we express informally as the sequence

$$\text{Galilei} \rightarrow \text{Einstein} \rightarrow \text{(anti-)deSitter} \rightarrow \text{conformal}. \tag{14}$$

A. Newtonian Kinematic: the Galilei Group

The Galilei group is the group of isometries[7] of the space-time $\mathbb{R} \times \mathbb{R}^3$, characteristic of Newtonian physics, where the first component of this space-time stands for the time direction.

The (proper[8]) Galilei group $\mathcal{G}$ is defined as the group with elements of the form

$$g = (b, \vec{a}, \vec{v}, R)\,, \tag{15}$$

[7] There are some ambiguities in the definition of the Galilean group as an isometry of a space-time. These ambiguities come from the fact that even though one is given a fixed underlying space metric together with a time direction, one may readily seen, after Milne as pointed out by Duval [12], that for different classes of connections there are different isometries groups we may consider. In hindsight, the Galilean group we are interested in could be seen as the one obtained via the Inönü-Wigner contraction of the Poincaré group. This choice breaks the ambiguity by explicitly pinpointing the Galilean group we are talking about. We call the Galilean group obtained in this manner by Galilei group.

[8] The adjective proper means that one is not considering parity (equivalently, spatial-inversion or reflection through the origin).

where b is a time translation; R a three-dimensional rotation; $\vec{a} = (a_x, a_y, a_z)$ a three-dimensional spatial translation; and $\vec{v} = (v_x, v_y, v_z)$ is a Galilei transform (or boost). A Galilei boost transports a spatial frame to another spatial frame uniformly moving with relative velocity $v = |\vec{v}|$ with respect to the previous one.

The transformation g in (15) acts on a general space-time point (representing an event) labeled by $(t, \vec{x})$ as

$$\begin{aligned} (t, \vec{x}) &\mapsto g \cdot (t, \vec{x}) \equiv (t', \vec{x}'), \\ \vec{x}' &= R \cdot x + \vec{v}t + \vec{a}, \\ t' &= t + b. \end{aligned} \tag{16}$$

The successive actions of two distinct elements g and g' provides the multiplication rule

$$g'g = (b', \vec{a}', \vec{v}', R')\ (b, \vec{a}, \vec{v}, R) = (b' + b, \vec{a}' + R'\vec{a} + b\vec{v}', \vec{v}' + R'\vec{v}, R'R). \tag{17}$$

The identity element is given by $e = (0, 0, 0, 1)$ and the inverse of g is given by

$$g^{-1} = (-b, -R^{-1}(\vec{a} - b\vec{v}), -R^{-1}\vec{v}, R^{-1}). \tag{18}$$

This representation can be nicely encoded in terms of 5×5 matrices. The 4-vector $(t, \vec{x})$ is written as a 5×1 vector column by adding a line with entry 1. The action of the Galilei group on a space-time point is then

$$g \cdot (t, \vec{x}) = \begin{pmatrix} R & \vec{v} & \vec{a} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} R \cdot \vec{x} + \vec{v}t + \vec{a} \\ t + b \\ 1 \end{pmatrix}. \tag{19}$$

The Galilei group admits the maximal Abelian subgroup $\mathcal{U}$, generated by spatial translations and the Galilei boosts. The quotient $\mathcal{G}/\mathcal{U}$ (the group of classes of equivalence) is not a simple group: it still contains a (maximal) Abelian subgroup $\mathcal{T}$ generated by time translations. The factor group $(\mathcal{G}/\mathcal{U})\,/\mathcal{T}$ is the simple group $\mathcal{R}$ of three-dimensional rotations. Therefore, the associated Lie algebra of the Galilei group is factorized as

$$\mathfrak{g} = (\mathfrak{r} \ltimes \mathfrak{t}) \ltimes \mathfrak{u} \equiv (\mathfrak{so}(3) \ltimes \mathbb{R}) \ltimes \mathbb{R}^3. \tag{20}$$

CORRECT FORMULA

Ten parameters are needed to describe the Galilei group: one for time translations, three for spatial translations, three for Galilei boosts and three for three-dimensional rotations.

A unitary representation of the Galilei group may be obtained by the method of induction (or method of "little groups"). For such representations we necessarily have to add a central term that may be associated to inertial mass. This fact has deep physical consequences which hints at the stability of particles in quantum mechanics. If the central extension is considered, the number of parameters needed to describe the Galilei group increases to eleven, where the inertial mass is recognized as the eleventh parameter.

The Lie algebra of the Galilei group admits then ten generators (H, P_i, C_i, L_{ij}), $i, j = 1, 2, 3$, where H (also known as Hamiltonian) generates time-translations, the three P_i (also known as momenta) generate spatial-translations, the three C_i generate the Galilei boosts and the three $L_{ij} = -L_{ji}$ generate spatial rotations. Their non-zero commutators are

$$\begin{aligned} &[C_i, H] = iP_i, && [L_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{kj}P_i), \\ &[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} - \delta_{il}L_{jk} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}), && [L_{ij}, C_k] = i(\delta_{ik}C_j - \delta_{kj}C_i). \end{aligned} \tag{21}$$

In the case of the central extension of the Galilei group, the additional generator, M, associated with the inertial mass of a particle commutes with all other generators. Moreover, the commutator between the spatial-translations and the Galilei boosts are no longer zero, being modified to $[P_i, C_j] = i\delta_{ij}M$.

The Galilei group depicts the kinematic symmetries of the non-relativistic free particle. The later obeys the equations of motion $\ddot{x}_i = 0$, $i = 1, 2, 3$. These equations may be derived from a variational principle involving an action invariant under the Galilei group of transformations. This action is defined as the integral $S = \int dtL$ of a Lagrangian [9] $L = \frac{m}{2}\dot{x}_i^2$. A variation of trajectory leaves the action invariant if and only if the Lagrangian is invariant up to a total derivative. This is the case here, namely [10]

$$L = \frac{m}{2}\dot{x}_i^2 \to L' = L + \frac{d}{dt}\left(mx_iv_i + \frac{m}{2}v_i^2 t\right). \tag{22}$$

[9] The Lagrangian may be seen as the difference of kinetic energy and potential energy (zero in this case). The real trajectories are obtained by minimizing S.

[10] This total derivative is intimately related to the need of the central extension of the Galilei group. This information is crucial for the quantization of the non-relativistic free particle.

B. Special Relativity: Poincaré Group

The Poincaré group is the group of isometries of the Minkowski space-time. This is a flat 4-dimensional space-time with the (metric) line element

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{23}$$

with c being the speed of light in the vacuum. It includes the space-time translations and rotations, together with their combinations.

The space-time translations generate its maximal Abelian subgroup $\mathbb{R}^{1,3}$. The factor (group of classes of equivalence) $\mathcal{L} = \mathcal{P}/\mathbb{R}^{1,3}$ is known as the (homogeneous) Lorentz group. This is a simple group. Therefore, the Lie algebra $\mathfrak{p}$ of $\mathcal{P}$ may be written as the semi-direct product (compare with (20))

$$\mathfrak{p} = \mathfrak{l} \ltimes \mathbb{R}^{1,3}, \tag{24}$$

where $\mathfrak{l}$ stands for the Lie algebra of $\mathcal{L}$. The Lorentz group $\mathcal{L} = SO(3,1)$ comprises the space-time rotations, which combine three-dimensional spatial rotations with Lorentz boosts. It is a subgroup of the Poincaré group and the latter is also known as the inhomogeneous Lorentz group.

We will focus on the component connected to the identity: the *proper orthochronous* Poincaré group (that we will also write as $\mathcal{P}$), which does not include the discrete transformations of time-reversal and of parity. Its general element may be written as

$$g = (a, R) \in \mathcal{P}, \tag{25}$$

where $a = (a_0, a_1, a_2, a_3)$ is a 4-translation and R a four-dimensional rotation belonging to the Lorentz group. The action of $g \in \mathcal{P}$ on a point $x = (x_0, x_1, x_2, x_3)$ of Minkowski space-time is given by (compare with (16))

$$x \mapsto g \cdot x = x' = R \cdot x + a. \tag{26}$$

The multiplication rule of the Poincaré group is (compare with (17))

$$g'g = (a', R')(a, R) = (a' + R'a, R'R). \tag{27}$$

The identity element is given by $e = (0, 1)$. The inverse of an element g is given by (compare with (18))

$$g^{-1} = (-R^{-1}a, R^{-1}). \tag{28}$$

This representation of $\mathcal{P}$ can be encoded in terms of 5×5 matrices: the four-dimensional space-time point $x = (x^\mu)$ is written as a 5×1 vector column. The action of $\mathcal{P}$ is given by (compare with (19))

$$g \cdot (t, \vec{x}) = \begin{pmatrix} R^\mu{}_\nu & a^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^\nu \\ 1 \end{pmatrix} = \begin{pmatrix} R^\mu{}_\nu x^\nu + a^\mu \\ 1 \end{pmatrix}. \tag{29}$$

We need ten parameters to describe the Poincaré group: four for the space-time translations and six for the Lorentz group (three for spatial rotations and three for the Lorentz boosts). The Lie algebra of $\mathcal{P}$ admits ten generators, $(P_\mu, J_{\mu\nu})$, $\mu, \nu = 0, 1, 2, 3$. The four P_μ (also known as 4-momenta) generate the space-time translations and the six $J_{\mu\nu} = -J_{\nu\mu}$ generate space-time rotations (three spatial rotations and three Lorentz boosts). The commutators are (compare with (21))

$$[P_\mu, P_\nu] = 0, \qquad [J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\rho\nu} P_\mu), \tag{30}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma}). \tag{31}$$

The Lorentz group is associated with the kinematical symmetries of a relativistic free particle. The corresponding equations of motion are $\ddot{x}_\mu = 0$, where the dot here means a variation with respect to a (proper time) parameter, so that one of the equations is $\ddot{x}_0 \equiv \ddot{t} = 0$. These equations of motion may be derived from an action which, like its Lagrangian, is invariant with respect to the Poincaré group.

The above kinematics were discussed by A. Einstein in one of his famous 1905 works, where he proposed a kinematics covariant under the symmetries of the electromagnetism. Before A. Einstein there was an astonishment among physicists due to the fact that the symmetries of the Maxwell equations describing electrodynamics (i.e., the Poincaré group) differed from those of the Newton-Galilei kinematics, i.e., the Galilei group.

C. From Galilean Kinematics to Special Relativity

The Galilei group $\mathcal{G}$ admits the subgroup $ISO(3)$, generated by the spatial rotations $SO(3)$ and spatial translations $\mathbb{R}^3$. This group is not simple and it admits a natural Inönü-Wigner extension, with parameter $1/c$: this gives the Lorentz group $SO(3,1)$, which is stable, i.e., admits no further similar extension.

This provides the extension of the full Galilei group to the Poincaré group, i.e., from Newtonian kinematics to special relativity. The Poincaré group is however not simple and the process may be continued.

D. Cosmology: $SO(3,2)$ and $SO(4,1)$

We now describe the remaining two simple groups of our list of kinematic groups (they are simple in the sense that they are not a semi-direct product of other subgroups): the anti-deSitter (adS) group $SO(3,2)$ and the deSitter (dS) group $SO(4,1)$. They act as groups of isometries of some respective four-dimensional space-times of constant but not zero curvatures.

The Lie algebras of each of these two groups admit ten generators $J_{ab} = -J_{ba}$, $a, b = 0, 1, ..., 4$, with commutators (compare with (20))

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad}), \tag{32}$$

where for dS

$$\eta_{ab} = \mathrm{Diag}(-1, +1, +1, +1, +1), \tag{33}$$

and for adS

$$\eta_{ab} = \mathrm{Diag}(-1, -1, +1, +1, +1). \tag{34}$$

The dS (adS) group is associated with the symmetries of the relativistic *** massless ??? *** Êparticle moving in the four-dimensional deSitter (anti-deSitter) space-time, with a constant positive (negative) curvature. The deSitter space-time is simply-connected. The equations of motion are of the form $\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\, \dot{x}^\nu \dot{x}^\rho = 0$, where dot here means a variation with respect to proper time and $\Gamma^\mu{}_{\nu\rho}$ are a set of functions associated with parallel transports on the underlying curved space-time. The action, and also the Lagrangian, from which these equations of motion derive are also invariant with respect to the dS group.

E. From Special Relativity to Cosmology

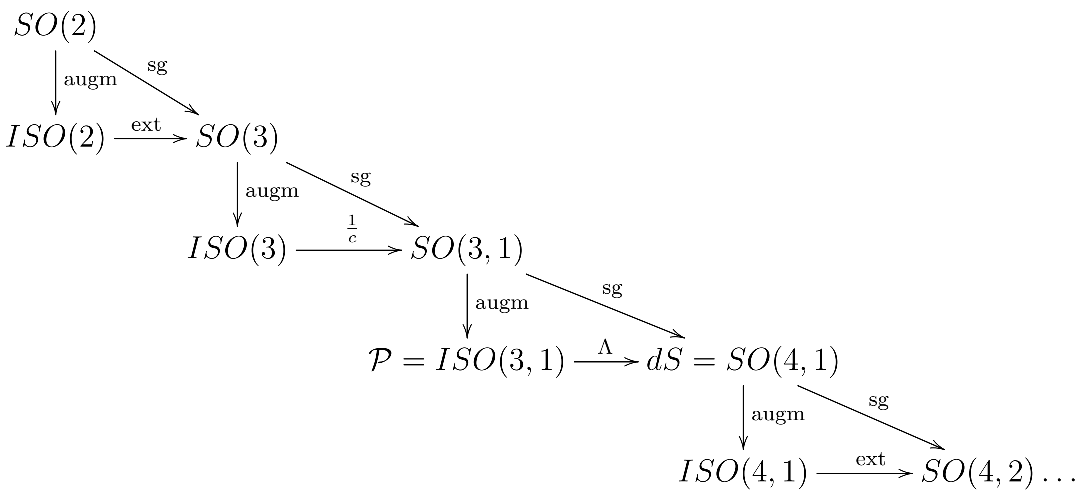
Like from $SO(2)$ to ISO(2)=E(2), the natural augmentation of $SO(3)$ is $ISO(3)$: the extension from the *isotropies* of Euclidean space to its *isometries* by adding the translations to the rotations. In the same way, $\mathcal{P} = ISO(3,1)$ is the natural augmentation of $SO(3,1)$: it completes the space-time translations with the space-time rotations.

Similarly, $\mathcal{P}$ admits a family of natural extensions with a parameter Λ called the cosmological constant. A positive value gives dS $= SO(4,1)$, while a negative Λ gives AdS $= SO(3,2)$. The limit value $\Lambda \to 0$ corresponds to their common contraction to the Poincaré group $\mathcal{P}$. Both dS and AdS groups are stable, in the sense that they admit no further similar extension.

Interestingly, the process may be continued. The augmentations of the dS and AdS groups (by adding translations) give respectively $ISO(4,1)$ and $ISO(3,2)$. Through the same process, both admit a common extension under the form of the *conformal group* $SO(4,2)$. The conformal group admits ten generators of the kinematic group, augmented by five new generators, one for *scaling transformations*, and four generating the so called *special conformal transformations*.

This group plays an important role in physics. For instance it preserves electromagnetism. As proposed initially by Weyl [31] it may constitute the symmetry group of a *conformal theory of gravitation*.

Finally, we may resume the previous results through the following diagram, with a categorical flavor,



The oblique arrows indicate subgroup inclusion denoted by sg. This diagram may be continued, but without straightforward applications to physics. A similar version of this diagram applies where the dS $= SO(4,1)$ and $ISO(4,1)$ are replaced by AdS $= SO(3,2)$ and $ISO(3,2)$, respectively.

Furthermore, the groups in the above series act as isometries of the space-times indicated in the diagram below, where the symbol ie stands for isometrical embedding.

$$
\begin{array}{ccccccccc}
SO(2) & \xrightarrow{\text{sg}} & SO(3) & \xrightarrow{\text{sg}} & SO(3,1) & \xrightarrow{\text{sg}} & SO(4,1) & \xrightarrow{\text{sg}} & SO(4,2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^2 & \xrightarrow{\text{ie}} & \mathbb{R}^3 & \xrightarrow{\text{ie}} & \mathbb{R}^{3,1} & \xrightarrow{\text{ie}} & \mathbb{R}^{4,1} & \xrightarrow{\text{ie}} & \mathbb{R}^{4,2} \\
 & & & & & & \uparrow{\scriptstyle \text{ie}} & & \\
 & & & & & & dS & &
\end{array}
$$

The notations $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^{3,1}$, dS stand respectively for Euclidean plane, Euclidean space, Minkowski space-time and deSitter space-time. The lower vertical arrow indicate that dS is embedded in $R^{4,1}$.

F. Remarks on the Quantum Domain

So far we have focused on transitions among kinematical symmetries. A similar discussion may account for the theory change leading from the classical to the quantum domain.

Classical dynamics (for instance of the non-relativistic particle) may be described by a Hamiltonian formalism in a phase space Γ. This is a symplectic manifold, whose canonical (position and momentum) coordinates generate a commutative algebra of functions on Γ. This is the Poisson algebra of classical observables. The process of quantization correspond to the replacement of this algebra by a non-commutative algebra together with some extra conditions: the algebra of quantum observables, seen as operators acting on an Hilbert space.

Mathematically, this can be accomplished by an *algebra deformation* procedure. The latter associates to the Poisson algebra of classical observables a parametrized family of non-commutative algebras. The quantum algebra of interest is then obtained with the value $\hbar$ of the parameter. The deformation procedure applied to Poisson algebra bears some similarities with the extension of Lie algebras (see, e.g., [28]).

The deformation can be obtained through the definition of a new product, the *star-product* $(f, g) \to f \star g$. This defines a *quantization map* which associates to each function f (classical observable) an operator $\widehat{f}$ (quantum observable) such that

$$\widehat{f}(g) \overset{def}{=} f \star g.$$

The deformation can be seen as a replacement of the original algebra by the new algebra of such operators. Such Poisson algebra deformation is considered one of the general procedures for quantizing a system. The non commutation resulting from the deformation is at the origin of the quantum uncertainty relations.

G. Non-commutative geometry

The diagram of isometries above (previous page) allows a geometrical interpretation of the procedure of algebra extension: by considering each algebra in the chain as the isotropy algebra of a pseudo-Riemannian manifold, it provides a chain of generalizations of such manifolds. Although very important in physics, this is not the end of the story. The deformation quantization seen in the previous section may be seen as a transformation of a classical phase space (manifold) into a *non-commutative space* which is not a manifold.

This may be seen as a special and archetypal case of *non-commutative geometry* which provides a generalization of manifolds under the form of new geometrical entities called *non-commutative spaces*. The latter are not made of points and are in some sense discrete. Their coordinates do not commute. The non-commutative geometry proceeds through a duality between algebra and geometry generalizing the Gel'fand duality. The latter establishes a one-to-one correspondence[11] between (compact) topological spaces and commutative (unital) algebras (the algebras of functions on them). The generalization of this correspondence to non-commutative C-star-algebras precisely generates non-commutative geometry [9] which associates a (possibly) non-commutative space to a C-star algebra. This procedure has the geometrical interpretation of upgrading a manifold (a commutative space) to a non commutative space. Quantization may be seen as a particular case when applied to the phase space manifold.

Let us mention that this procedure is also applied to space-time, with the goal of constructing a new physics in the frame of a *non-commutative space-time* [25]. This intends for instance to give a phenomenological description of the effects of quantum gravity which are thought to destroy the manifold structure of space-time at small scales.

Another original possibility, developed by A. Connes and collaborators [7, 8], considers matter fields living in an *internal* geometrical space. This is already the common view in classical field theories but this internal space is described here as a discrete, non-commutative

[11] This is conveniently analyzed in the frame of *category theory*, where this correspondence appears as a categorial equivalence established through adjoint functors. [21][22]

space. The geometrical framework of the theory is provided by the product of the four-dimensional (commutative) space-time manifold by this non-commutative internal space. Application of a *spectral action principle* led Connes and his collaborators to a pure geometrical derivation of the physics of both standard model and gravity.

H. Summary: Contraction versus Extension

The Inönü-Wigner extension generates a progression from Newton kinematics to (anti-)deSitter and conformal kinematics. The opposite Inönü-Wigner contraction provides the way back. An important issue concerns the physical meaning of the parameter involved in the process. This may suggest a connection with other approaches of theory change based on constructs and logic [13]. A physical interpretation of the parameter requires the use of the representation theory of the underlying symmetry group.

Let us exemplify this point. In Newtonian physics, distances are endowed with a notion of dimensionality: the result of a distance measurement is a ratio between a distance and its unit (i.e., a meter). Similarly, time-durations are also endowed with a notion of dimensionality: the result of a time measurement is a ratio between a duration and its unit (i.e., a second). Newtonian physics is thus characterized by two types of metric dimensionalities, time and space.

Playing the game of Inönü-Wigner contraction/extension leads to the Poincaré group which, in its *fundamental representation*, acts on the Minkowski space-time by preserving its metric element (23). The resulting measuring procedures require a synthesis between the two original (Newtonian) types of units. It is obtained through a conversion factor between them, which identifies with the extension parameter $1/c$ (inverse of the speed of light) as it appears in (23).

*** this has been officialized ... ***

The Newtonian limit $1/c \to 0$ applies in the small (compared to c) velocity approximation, where the conversion becomes irrelevant for practical use, so that distinct distance and time units are reintroduced.

***A similar discussion applies to the contraction from AdS (or dS) to the Poincaré group. The (anti-)deSitter space-time is defined in terms of its non-zero curvature, the cosmological constant Λ which identifies with the deformation parameter. It provides a specific unit $\Lambda^{-\frac{1}{2}}$ identified to the curvature radius of that space-time, on which the (A)dS group acts as isometry group.

Therefore its generators can be rescaled using the curvature as parameter. This at one shot provides a physical meaning for the generators of the group and sets up the stage for the Inönu-Wigner contraction. It is now a simple matter of taking the limit at the level of the rescaled Lie algebras as the curvature reaches zero to obtain the Poincaré Lie algebra. ***

This last example in its 3-dimensional avatar was explicitly worked out in section III-A-1.

V. DISCUSSION AND CONCLUSION

We have analyzed theory change under the perspective of an improvement of the characteristic symmetries. This exhibits in our opinion the inner-workings of the process and allows us to address the problem of theory incorporation — and hence of scientific discovery [20] — in terms of group inclusion.

The so called *constructs* [IS IT AN USUAL TERM ?] (also known as observables) which characterize physical theories — like energy, momentum, spin, etc. — are basically obtained from the theory of representation of the symmetry involved. This is where the connection with the Bargmann-Wigner program enters into stage. Now, as emphasized by Inönü in [17] – his personal recollection of his joint work with Wigner on the contraction of groups, the attempt to apply correspondence limits – theory change – at the symmetry level directly to the representations had "became incomprehensible": [I DO NOT UNDERSTAND THAT SENTENCE]

> "... the original programme proposed to me by Wigner was completed and I started to write the paper on the Galilei representations. But a question remained: How is it that, the true representations of the Poincaré group have a physical meaning while those of the Galilei group do not? Or, in other words, how does the physical meaning disappear when one goes over from the Poincaré group to the Galilei group? We thought that at least a partial answer could be obtained by looking at the limits for infinite light velocity of the specific unitary representations of the Poincaré group obtained by Wigner. The idea was to add

an appendix to our Galilei paper, giving the results of this limiting process.

However, when I tried to take the limits of the unitary representations of the Poincaré group, the outcome became incomprehensible. The limiting process gave a finite answer in some cases, but vanished altogether in other cases. After we struggled for a couple of weeks without obtaining consistent results, Wigner had the bright idea of separating the problem into its essential components. He said: 'Let us first look at the limit of the group, understand what happens there, and then consider the limits of the representations.' This approach gave the clue for solving our difficulties. (...)" [**Excerpt from [17]**.]

We estimate that the precision of our mathematical procedure, describing theory change via symmetry improvement, is not achieved in other exclusively logic approaches like for instance the semantic one. The same conclusion is given by Halvorson [16] when he states that despite their adequateness, the logic approaches are too general to usefully characterize specific disciplines.

As far as physical theories are concerned, the present work has shown the heuristic power embodied by the concept of symmetries – here presented in the framework of group theory – for the discovery of new theories and the setting of their validity boundaries. Therefore we may assert that the mathematical formulation together with a physical interpretation, enables one to determine a physical theory and, in a sense, indicates a logic of discovery where concepts directly emerge from the mathematical basis. In the present case this basis stands on the symmetries of the physical theory.

The analysis of the mathematical structure in the present work paves the way to a characterization of physical theories of a free particle. It also indicated a procedure for the discovery of new theories of similar type and it sets up unambiguously their context of validity.

Appendix A: The Wigner-Bargmann Program

Our assertion that a physics theory, at least those for non-interacting particles, can be characterized in terms of kinematic symmetries and their representations is borrowed from the Bargmann-Wigner program. This program provides a bridge between the proposed characterization and other attempts to define a physical theory by means of its constructs.

The Bargmann-Wigner program consists of a procedure to tackle the following problem. Given a unitary irreducible representation (unirrep) – quantum domain – of a kinematic group – symmetry –, one can construct a differential equation and its positive-frequency solutions which manifestly transform under the given unirrep.

In the late 40's, Bargmann and Wigner considered this problem in respect with some unirreps of the Poincaré group (at least those that are physically relevant). They obtained the known relativistic wave equations of Klein-Gordon and Dirac. Later on, the program was completed for other unirreps. For the case of Galilei groups, the program initiated by Wigner and Inönü, which inspired them into their notion of contraction, was later completed by Lévy-Leblond.

The steps of the Bargmann-Wigner program may be summarized as follows [4, 5, 32]:

1. Choose a unirrep of the kinematic group. In principle, one can work not necessarily with a unirrep but with a symplectic irreducible representation of the kinematic group. This choice of representation would then provide the corresponding classical Hamiltonian equations from which corresponding geodesic equations can be obtained. We continue the list of steps for the quantum aspect of the procedure only.

2. Write wave functions on the underlying space-time manifold taking values on the chosen unirrep of the previous item.

3. Write a system of differential equations on the space-time manifold for which the written wave function is a solution. This differential equation may be written as an algebraic equation by Fourier transforming it.

4. Look at the physical degrees of freedom. Some components of the wave function may be unphysical like for instance the longitudinal oscillations of the massless relativistic wave functions. Such components should vanish and only the real physical degrees of freedom would be valued in the unirrep of item 1.

We refer to the nice reference [5] for explicit examples of this program like that of the Klein-Gordon equation.

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