Categories and Physics

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Categories

- New visions on geometry and generalizations (topos...)
- Discrete geometric entities: causal sets, simplicial sets, spin networks, spin foams...
- Correspondences
  - geometry - algebra
  - geometry - logic
  - continuous - discrete
- Relational view: relational physics
- Generalization of mathematical structures
- New fundations of mathematics
Outline

Category: definition
Cobordisms
Functors
Tensor category
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topos for quantum Physics (Isham)
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(The spectral presheaf)
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The sub-object classifier
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quantum gravity
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Category: definition

*ensembles* are made of elements (objects)

*categories* are made of *objects*

and *arrows* (= morphisms) between them.

(an arrow can be invertible or not)

**Only rules**: associativity; one identity morphism for each object

(invertible)

**Richer structure**: one dimension more!
isomorphism, i.e., an arrow $f$ which has an [uniquely determined] inverse $g$ such that the compositions $fg$ and $gf$ are identities.

**Initial object** $i$: there is an arrow $i \to c$ for each object $c$;

**Final object** $f$: there is an arrow $c \to f$ for each object $c$;

In **Set** (The basic example),

Objects are sets, arrows are maps between them;

Initial object $\emptyset$; Final object $1 = \{\ast\}$: the singleton
The basic example: **Set**

Objects are sets, arrows are maps between them; Usually, an ensemble is defined by the properties of its elements. In categorical view, a set is defined through its relations to other sets. example::

each set $S$ is an object in **Set**.
An **element** of $S$ is defined as an arrow from $\{\ast\} \to S$.
A **product law**: an arrow $S \times S \to S$.
An **automorphism**: an invertible arrow $S \to S$.
A **subset** of $S$: an arrow $S \to \Omega = \{0, 1\}$.

... All these definitions generalize (or not) to other categories
# Basic categories

In **Set**: objects are sets, arrows are functions

In **Cat**: objects are categories (generalized sets), arrows are functors (generalized functions)

ex.: **Vect**: vector spaces / morphisms = linear appl.

ex.: **Hilb**: Hilbert spaces / bounded linear maps...

groups, manifolds,...

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An object has no elements, or any sort of internal structure defining its properties. What matters (its properties) are its morphisms to and from other objects: things are described, not in terms of their constituents, but by their relationships to other things. Relational view $\simeq$ the way physics describes the world?!
Different views

▶ See a group \( G \) as a category \( \hat{G} \) with a single object; and all morphisms invertible (elements of \( G = \) the morphisms in this category).

Ex.: a causal set is a cat with at most one morphism between 2 objects.

▶ A group is an object in the cat of groups

▶ A group is a group-object in the category Set. (” group-object ” means an object which obeys some list of properties)

What are group-objects in an other category?

INTERNAL / EXTERNAL view
The category of Cobordisms

a category for general relativity: Dynamical problem: find spacetime, given initial and final conditions (= initial space and final space)

Cob (or nCob):

\[ M : \Sigma_1 \rightarrow \Sigma_2; \partial M = \Sigma_2 \cup \Sigma_1^*, \]

objects = the \((n-1)\)-dim. manifolds \((\simeq \text{spaces})\);
arrows = the n-dim manifolds between them \((\simeq \text{space-times})\).
Cob is a tensor category

Figure: Tensor product in Cob (from John Baez)
Cob has duality properties.

Figure: The adjoint of a cobordism (from John Baez)
Adding metric properties (Riemannian cob., Lorentzian cob., etc.) provides the natural expression of general relativity in the categorical framework.

(There are discrete versions, with manifolds replaced by combinatorial entities like graphs:

→ spin-networks and spin-foams, at the basis of QGR)

**Cob** has properties different from **Set**. But analog to other categories adapted to quantum physics! (see below) (unexpected ?; miraculous ?).

A **field theory** is a functor between those types of categories. (the explanation why they work ?).

→ search for quantum gravity is as functors of this type ?...
Functors

= maps between categories, preserving the structure:
  Objects → objects
  Morphisms → morphisms

(a functor is a morphism in the category Cat.)

▶ Ex.: A representation of a group $G$ to a category $C$ is a functor $\hat{G} \to C$.
It selects an object $X \in C$ and a group homomorphism $G \to Aut(X)$, the automorphism group of $X$.

▶ The usual case of a linear representation is
  a functor $\hat{G} \to Vect$ (the cat of vector spaces).

▶ In Physics, a TQFT is a functor $\text{Cob} \to \text{Hilb}$ (manifolds to Hilbert spaces).
Other field theories generalize that (and QGR try also)
Tensor category (= monoidal category):

has a tensor product (= monoidal product) combining two objects into a new one:

\[ \otimes : C \times C \rightarrow C, \]

(it must obey some axioms which express, e.g., associativity),

and an unit object (= monoidal unit) \(1_C\) satisfying

\[ 1_C \otimes a \simeq a \otimes 1_C \simeq a, \]

[ If the isomorphisms are identities, the monoidal category is strict. ]

- In \textbf{Set}, the Cartesian product is a tensor product (with more properties which allow to call it Cartesian product).
- In \textbf{Vect} or \textbf{Hilb}, usual tensor prod; at the basis of categorical expression of quantum physics

In \textbf{Cob}, \textbf{Vect}, \textbf{Hilb}..., the tensor product is not Cartesian!
the tensor category $\text{Rep}(G)$

$\text{Hilb}$, the category of Hilbert spaces has special interest for [quantum] physics.
Also $\text{Rep}(G) =$ category of the representations of a group $G$.

- **objects** are the [direct sums of a finite number of] irreducible representations of $G$.
- **Morphisms** are linear maps which intertwine the group actions.
- A representation is a functor (to $\text{Vect}$ or $\text{Hilb}$):
  $\Rightarrow \text{Rep}(G)$ is a category of functors (very interesting).

(This can be generalized to Hopf algebras, quantum groups...)

**Perfect for quantum physics** : a typical example is Feynman diagrams
→ **Feynmanology** (Crane, Baez...):
Feynman diagrams as categories

- objects = Hilbert spaces
  (Wigner: particles = irr unitary representations for Poincaré group, )
- morphisms = bounded linear operators between them (with usual composition)
- usual tensor product (not a Cartesian product !)
- the categorical view expresses all quantum physics as a generalization of F diagrams; and QGR tries to do the same.

Figure 1 - Exemples de diagrammes de Feynman pour le processus de diffusion entre deux électrons. Les lignes droites représentent les électrons (e) et les ondulées les photons (γ). Les flèches montrent le flux de la charge électrique.
Hilb

(objects = Hilbert spaces; morphisms = bounded linear operators between them, with usual composition)

Usual tensor product (not a Cartesian product !)

Duality (of objects and morphisms)

A $\dagger$-functor (= \textsc{dag functor} = adjoint functor) is a contravariant endofunctor $C \to C$ which reduces to identity on objects, and such that $\dagger \circ \dagger = \mathbb{1}$.

(The transformed $f^\dagger$ of a morphism $f$ is called its \textit{adjoint}.)

A $\dagger$-category (or \textit{dagger-category}) is a category equipped with a particular choice of $\dagger$-functor.
In the cat $\text{Hilb}$, a morphism (bounded linear map) $f: H \to J$ has adjoint $f^\dagger : J \to H =$ the unique map such that

$$\langle f(\phi) \mid \psi \rangle_J = \langle \phi \mid f^\dagger(\psi) \rangle_H.$$
Many common properties between **Cob** and **Hilb** (general relativity and quantum !),
(both are unitary $\dagger$-categories, *symmetric monoidal categories*
not shared with **Set**. !
”Our intuition is more or less modeled on the structure of the
category of sets, this difference (between **Set** and **Hilb**) could
explain why our intuition does not apply to quantum physics. ”

Moreover, ” many of the ways in which **Hilb** differs from **Set** are
ways in which it resembles **nCob**! This suggests that the
interpretation of quantum theory will become easier, not harder,
when we finally succeed in merging it with general relativity. ”
Motivates the categorical approach to QGR ( L Crane).
A topological quantum field theory is a functor

\[ \text{TQFT} = \text{functor} \ Cob \rightarrow \ Hilb \]

- each manifold (representing space) \( \rightarrow \) an Hilbert space;
- each cobordism (representing spacetime) \( \rightarrow \) an operator between the Hilbert spaces (preserving composition and identities).

\[ \rightarrow \] reformulations of quantum field theory in terms of categories under the name of \textit{algebraic QFT} and further extensions like \textit{functorial QFT}.

algebraic QFT

An algebraic QFT = a covariant functor $\mathcal{T} \rightarrow \mathcal{A}$, where

$\mathcal{T}$ is a category of geometrical or topological type: objects are topological spaces, possibly with some additional structure (preserved by the morphisms)

$\mathcal{A}$ is a category of algebras, also with prescribed structure (like C-star algebras) describing the observables.

Very general scheme for physics!
Usual (non-relativistic) quantum mechanics enters in this scheme (as a (1+0)-dimensional QFT). → the categorical quantum mechanics approach of Abramsky and Coecke
Categorical quantum mechanics approach of Abramsky and Coecke


Samson Abramsky and Bob Coecke, Categorical Quantum Mechanics, in Handbook of Quantum Logic and Quantum Structures vol II. Elsevier, 2008, arXiv:0808.1023v1,

Figure: preservation of inner product

and now: the category of quantum computation…
Topos

Special categories called topoi

- A topos is a cat defined by a collection of axioms. finite limits and colimits, power objects (xxx), a subobject classifier.

- new ways for interpretation of quantum Physics: geometry - algebra - logic - propositions...

- A topos is a category which generalizes the cat of sets. Set is the simplest topos, a model for other topos: generalized sets, generalized functions.

- Every topos has a geometrical interpretation and a (non Boolean) logical one: a language of propositions: (Set describes the classical propositions; a convenient topos describes the quantum ones)

→ topos view on quantum physics
→ topos view on causal set
Internal language

In **Set**, an algebraic structure (e.g., group...) is an object which obeys some internal rules.

A topos \( T \) has internal language: An object \( T \) which obeys the rules of (algebraic) structure \( S \) is a \( S \)-object.

(ex: an object in \( \text{Grp} \) (internal lang) is a group (external) / a group-object in \( \text{Grp} \) (internal) is a commutative group (external)...

A classical system \( = \) some structure (implicitly in the topos **Set**).

Corresponding quantum system \( = \) same structure internal to a topos \( T_Q \).

a classical observable has (for a state) a value in \( \mathbb{R} \)

a quantum observable has (for a state) a value in \( \mathcal{R} \): the real-number-object in \( T \)

Transpose the same structure in an other topos \( \rightarrow \) the quantum version of the Physical system:

classical \( \rightarrow \) quantum; gravity \( \rightarrow \) quantum gravity.
**ex.: Site and Topos**

A fundamental example (in topology)  
**Site** $O(X)$ of a topological space $X$  
- $= \text{cat of its open sets, with inclusions as morphisms.}$  
- The set $X$ is a terminal object.  
- The empty set is an initial object.  

A **presheaf** over $X$ is a functor $O(X) \to C$,  
($C$ is a category; generally **Set**).  

Presheaves are encountered everywhere in Physics. Ex: A fiber bundle (basic structure in gauge theory) is equivalent to the presheaf of its local sections! And presheaves form topos  

The category of presheaves over a site is its **topos** (Grothendieck): central object of study!  
Every (abstract) topos may be interpreted this way, as the topos of some site.

A quantum system is represented by the collection of *quantum observables* (= operators acting on an Hilbert space of states.)

They form a *non-commutative algebra* $\mathcal{A}$.

In classical physics, an observable assigns a value to a state. **Not in quantum physics.**

*Replace* **Set** by a topos constructed from $\mathcal{A}$.

$\rightarrow$ new geometry; information, knowledge, logic, propositions ...  

$\rightarrow$ a new logic (not the "quantum logic which is not distributive; but an intuitionist logic" truth values " are attributed to propositions which may be yes, no **OR SOMETHING ELSE**.)*
The context category

One starts from the non-commutative algebra $\mathcal{A}$ of \textit{observables} of a quantum system, seen as operators on $H$. A \textit{context} $V$ is a commutative subalgebra of $\mathcal{A}$. (a set of variables (observables) that are simultaneously measurable by observers). $\Rightarrow$ a classical point of view on the system.

With inclusion (to be seen as "coarse graining"), the contexts form the \textit{context category} $\mathcal{N}(H)$.

Replace the quantum point of view by the collection of all possible classical points of view (equivalent)
Presheaves

Contravariant functors from the context cat to Set form the category $T = \text{Set}^{\mathcal{N}(H)^{op}}$ which is a topos.

An object (≡a presheaf) is a map sending each context to a set. It is considered as a "generalized set".

A natural transformation between presheaves generalizes a function between sets.

Each presheaf $P$ may be seen as a collection $(P_V)_{V \in \mathcal{N}}$, where each $P_V$ is a set, with functions $P(i_{VV'}) : P_V \to P_{V'}$, being maps between them when $V' \subseteq V$.

One considers three particular objects in $T$:
- The terminal object sends every complex to singleton: $1_T : V \to \{\ast\}$
- The spectral presheaf $\Sigma$, also called the state object
- The sub-object classifier
The spectral presheaf

Def. The **Gelfand spectrum** $Sp(V)$ of an algebra $V$ is the set of all linear functionals $\lambda : V \to \mathbb{C}$ (such that $\lambda(1) = 1$) ($\cong$ the set of eigenvalues of the elements of $A$ seen as operators = possible outcomes of a measurement.)

Def. The **spectral presheaf** $\Sigma$ is the state object = is the [contravariant] functor which sends any context (algebra) to its Gelfand spectrum (a set)

$$\Sigma : \mathcal{N}(H)^{op} \mapsto \textbf{Set}$$

$$V \mapsto \Sigma(V) = Sp(V), \quad (3)$$

$$i_{V'}V \mapsto \Sigma(i_{V'}V) = \lambda_{|V'},$$ the restriction of $\lambda$ to $V' \subset V$.

$\Sigma$ is not like a set since it admits no (global) element.
Def. (particular cases of more general def)

- A **sieve** $\sigma$ of a context $V$ is a family of subalgebras of $V$ which is closed for inclusion, i.e., such that

  $$a \in \sigma, \ a' \subseteq a \Rightarrow a' \in \sigma.$$  

- The **maximal sieve** of $V$ is written $\downarrow V$.
- The **minimal sieve** of $V$ is the empty set.
- We write $\Omega_V$ the set of sieves on $V$. 
The **sub-object classifier** is an element of the topos: 
the functor $\Omega : \mathcal{V} \to \textbf{Set}$ (from the cat of contexts to $\textbf{Set}$) 
which sends any context $V$ to the set of sieves on $V$. :

$$\Omega : \mathcal{V} \to \Omega_{\mathcal{V}}$$

$$\Omega : i_{V \to V'} \to \Omega_{i_{V \to V'}} : \Omega_{\mathcal{V}} \to \Omega_{\mathcal{V}'}$$

$$\sigma \to \sigma \cap (\downarrow V').$$

▶ It has **global elements** called **truth value**

a global element is an arrow $\gamma : 1_T \to \Omega$, 
defined by a collection of sieves (sets) $(\gamma_V) \in \Omega_{\mathcal{V}}$,
such that $\gamma_{V'} = (\gamma_V) \cap \downarrow V'$ for $V' \subset \mathcal{V}$.

▶ The particular global element $\gamma_1$ defined by $(\gamma_1)_V = \downarrow V$ is 
called **totally true**.

▶ The particular global element $\gamma_0$ defined by $(\gamma_1)_V = \emptyset$ is 
called **totally false**.

▶ The truth values are ordered by inclusion. They form a 
**Heyting algebra** (→ new [intuitionist] LOGIC).
Topos and quantum

We reexpress the properties of the quantum system in $T$ rather than in $\text{Set}$. The sub-object classifier assigns a *truth value* to any proposition about the system. The truth values are ordered by inclusion. They form a *Heyting algebra* and define an *intuitionist logic*: the non Boolean logic of quantum proposition. (not the quantum logic).
A **causal set** (≡ *causet*) is a poset which is also *locally finite*:

$$|\text{Past}(x) \cap \text{Fut}(y)| < \infty,$$

(cardinality of the set).

= a discrete and approximate view of space-time.

Elements are (virtual) events

Partial order interpreted as causal relation.

(volume of a subset = the number of elements in it)

→ discrete version of Lorentzian geometry.

A representation of space-time limited to its causal structure
( at the basis of some quantization attempts...)

Fotini Markopoulou, *The internal description of a causal set: What the universe looks like from the inside*,

http://arxiv.org/abs/gr-qc/9811053v2
Topos and Causal set

an other example of topos in Physics
The causet is a cat $C$ with at most one morphism between 2 elements. The functors $C \to \text{Set}$, form the topos $\mathcal{T} = \text{Set}^C$. In particular

- **Past functor:** $p \to past(p) = \{ q \in C; q \leq p \}$

  
  $[ p \to q \iff p \leq q ] \to [ past(p) \to past(q) \iff (past(p) \subseteq past(q)) ].$

  
  this functor is called an **evolving set**.

  (not a set, because not an object in $\text{Set}$)

- **World functor:** $p \to C$, any arrow $\to \mathbb{I}$.

- **the terminal object** $1_\mathcal{T} : p \to \{*\}; (p \to q) \to \mathbb{I}$.
Time-till-truth value

Idea: In the ensemblist vision, an event Q is in the past of p or not: Yes or not (Boolean logic).
In the topos vision, there is a **time-till-truth value**:

- *true for* p *if* Q is in the past of p (≡ has already happened).
- But, during the future evolution of p, its past increases. At some moment of this evolution, Q may be in the past. This is indicated by a **truth value** between yes and not: **when Q will become past for** p **during its future evolution**.
- Q is **false** if it will NEVER be in the past of p during its future evolution.

This is obtained through a similar construction using co-sieves (dual to sieves), sub-object classifier: the latter gives a Heyting (not Boolean) algebra: intuitionistic logic (non excluded middle)
A **cosieve** $s$ at $p \in C \cong$ a remote future of $p$

= a closed subset of $\text{Max}_p \overset{\text{def}}{=} \text{Hom}(p, \cdot)$, such that $(p \to q) \in S$, $q \to q' \Rightarrow (p \to q') \in S$.

**Maximal cosieve** at $p = \text{Max}_p \overset{\text{def}}{=} \text{Hom}(p, \cdot)$.

The set of cosieves at $p$ is $\Omega_p$.

It corresponds to a set $\sigma \subseteq C$ such that $r \in \sigma \Rightarrow p \leq r$; $q \in \sigma$, $q \leq q' \Rightarrow q' \in \sigma$. 
The Sub object classifier is the functor $\Omega \in \mathcal{T} : p \rightarrow \Omega_p$. (For $p \leq q$, the map $\Omega_{pq} : \Omega_p \rightarrow \Omega_q : s \rightarrow s \cap \text{Max}_q$.

It has *global elements* = truth values. A truth value (=an arrow $S : 1 \rightarrow \Omega$) is defined by a collection of sieves ($S_p \in \Omega_p$). Each $S_p$ is a *time-till-truth value* at $p$.

The *totally true arrow* $T_{\text{true}} \in \Omega$ send any vent $p$ to its maximal sieve $S_p = \text{Max}_p$ the set of elements which a have already happened. The totally true *time value* at $p : \text{already}$.

The *totally false arrow* defined by $S_p = \emptyset : \text{never}$.
Motivations for quantum gravity (QGR)

- Unify Physics
- to find the microstructure of spacetime: continuous or discrete? In the latter case, how does the continuum approximation arise?
- How to describe the interior and the properties of a black hole? Understand its thermodynamical properties;
- Quantum cosmology is a branch of QGR, which may unveil the initial conditions and the remote past (primordial universe). Does space-time geometry make sense near the initial singularity?
General relativity

- **Goal**: to find the geometry of space-time, as a metric $g$ (in a given background differential manifold $M$) given an energy-matter content; and/or initial and final conditions.

- **Extended configuration space** = $\text{Met}(g) =$ space of all metrics living on $M$.

- **Covariance**: two metrics diffeomorphism-related represent the same physical solution.
  $\Rightarrow$ a solution = a class $[g]$ of metrics related by diffeos.

- **True configuration space of the theory** = superspace of $M$

  $$S = \{[g]\} = S = \text{Met}(M)/\text{Diff}(M)$$

  (complicated structure; not a manifold).

$\text{Diff}(M)$ is the group of diffeomorphisms acting on $M$ (one may restrict to those with compact support),
gravity = geometry (GR)

⇒ to quantize gravity = to quantize geometry.

What mean quantum geometry?
discrete character? (suggested by Einstein)

- **Discrete approaches**: fundamental structure is discrete and combinatorial: = a discontinuum.
  (e.g., causet, simplicial complexes...)
  continuous space-time emerging as a coarse grained approximation.

- The **canonical quantization** of general relativity leads to **loop quantum gravity** (LQG): also involves combinatorial structures: spin networks, spin foams... expressed in terms of graphs or complexes

Both approaches converge with the use of combinatorial structures and find natural expressions in the language of tensorial categories.
Discrete approaches to QGr

Start from a combinatorial structure at the beginning. (typically simplicial complexes).
This is in fact the original spirit of the spin-foam approach, and spin networks or spin foams are in duality with simplicial complexes or triangulations.

E.g., simplicial complex:
- The vertices are the objects,
- the edges are 1-morphisms,
- the triangles 2-morphisms etc.
Canonical quantum gravity

- *Introduce an arbitrary (artificial) space + time splitting of space-time.*
- *Covariance becomes constraints*
- *Use adapted (Ashtekar’s) canonical variables*
- *Quantize (canonically) following (Dirac’s method)*
- *Discrete [kinematical] states appear as spin networks (Penrose)*
- *They solve the constraints but the Hamiltonian one (dynamics).*
Spin networks

LQG $\rightarrow$ well defined Hilbert space; ON basis of spin networks

(Originally defined as combinations of loops)

This defines a quantum 3d Riemannian geometry:
Well defined geometric area and volume operators act on them.
Have discrete spectra; are diagonal in the spin network basis.

Discrete; at the Planck scale, the classical notion of space ceases to exist.
(This gives the kinetic Hilbert space of the theory. To find the dynamical one (i.e., to have a true QGR theory), one must solve the constraints.)
Spin networks were originally introduced by Penrose

(very like Feynman diagram).
They are the quantum states of 3-d riemanian geom.
A spin network is a one-dimensional oriented graph with carries labels (= decorations):

- on each edge, an irreducible unitary representation of SU(2) = a spin label the area eigenvectors;
- at each vertex, an intertwiner (= a tensor) mapping of the tensor product of incoming representations define the volume eigenvalues.

(May be generalized to other groups or quantum groups).
Dynamics and spin-foams

Spin-network states (\(=\) quantum states of space) represent the kinematical sector of LQG.
A dynamical [quantum] state of LQG may be seen as
- a spin network’s history (in the quantum language)
- a spin network’s world-sheet (in the relativistic language)
- a spin-network state solving the quantum motion equations. Those are the Hamiltonian constraint (which implements the dynamics) at the quantum level, a subject presently in progress.

Recent works attempt to describe it as a [linear combinations of] spin-foams: Spin-foams would describe a quantum 4-geometry in the same way that a quantum 3-geometry appears as a superposition of spin-networks.
A spin-foam is a tool allowing to calculate the transition amplitude between two quantum 3-geometries under the form of spin network states: analogy with cobordims.
to form a quantum history of the gravitational field

In fact spin foams offer a formulation for a large variety of theories.
III. SPIN-FOAMS OF SPIN-NETWORKS AND OF SPIN-NETWORK FUNCTIONS

A. Classical motivation
The motivation of the spin-foam approach to LQG is to develop an analog of the Feynman path integral. The idea of Rovelli and Reisenberger [2], is that the paths, should be suitably defined histories of the spin-network states. We address this issue in this section.

B. Spin-foams
1. Foams
By a foam we mean throughout this work an oriented linear 2-cell complex with (possibly empty) boundary. For the precise definition of the linear cell complex we refer the reader to [4, 23]. Briefly, each foam \( \kappa \) consists of 2-cells (faces), 1-cells (edges), and 0-cells (vertexes).

\[ \text{FIG. 3:} \]

The faces are polygons, their sites are edges, the ends of the edges are vertexes. Faces and edges are oriented, and the orientation of an edge is independent of the orientation of the face it is contained in. Each edge is contained in several (at least one) faces, each vertex is contained in several (at least one) edges.

The boundary \( \partial \kappa \) is a 1-cell subcomplex (graph) of \( \kappa \). A edge of \( \kappa \) is an edge of the boundary if and only if it is contained in only one face. Otherwise, it is an internal edge. A vertex of \( \kappa \) is a vertex of \( \partial \kappa \) if and only if it is contained in exactly one internal edge of \( \kappa \) (this is an important technical subtlety of the definition of a boundary). Otherwise, it is an internal vertex of \( \kappa \).

Definition A spin foam is defined as a 2-complex \( C \) with a two components coloration \( c = (a, b) : \)

- to each face (=triangle) \( f \) is associated an Hilbert space \( H_f \), an irrep of a group \( G \).
- to each edge \( e \) is associated an intertwiner \( b_e \). (a vector in an Hilbert space associated to \( e \), defined as the tensorial product of the Hilbert spaces of the adjacent faces (outgoing and ingoing dualized)

(some 2-complexes are associated with manifold triangulations ; but this is not the case for all of them)
Categorical view

*Category of oriented graphs* shows Analogy with *Cob*:

**Objects**: oriented graphs $\Gamma$ (which underly spin networks); play the role of spaces;

**Arrows**: 2-complexes $C : \Gamma \to \Gamma'$ with the initial and final graphs as boundary conditions: $\partial C = \Gamma \cup (\Gamma')^*$.

(underly spin-foams); play the role of space-time

- To each oriented graph $\Gamma$, a spin network attaches an Hilbert space $H_{\Gamma}$.
- To each 2-complex $C$ (underlying a spin-foam; instead of space-time), is assigned and a vector of $H_{\partial C}$.

→ a spin-foam model appear as a functor from the category of oriented graphs to that of Hilbert spaces (Crane, Rovelli..) :

$$Graphs \to \textbf{Hilb} : \Gamma \to H_{\Gamma}.$$ 

It obeys axioms similar to the [Atiyah] axioms of topological quantum field theory.
In LQG, $H_\Gamma = L^2 \left( \frac{\text{SU}(2)^L}{\text{SU}(2)^V} \right)$, the Hilbert space of functions which are square-integrable w.r.t. a well-defined measure.

The configuration variable for LQG is a SU(2) connection.

A spin network is originally defined as a functional of these connections, like a wave-function of the configuration variable in ordinary quantum mechanics.

Here, the functionals are obtained from the functions of $L^2 \left( \frac{\text{SU}(2)^L}{\text{SU}(2)^V} \right)$, labelled by spin networks.

The total Hilbert space is obtained by summing over all the graphs.

A **Feynman amplitude** is associated to the 2-complex (seen like a Feynman graph), which involves a sum over all possible colorings.

Then a [weighted] sum over all 2-complexes, keeping fixed “boundary data”, under the form of “initial” and “final” spin-network states (quantum 3-geometries) gives a partition function, to play the role of a path integral for quantum gravity. This summation over all spin foam configurations replaces a summation over the 4-geometries, keeping fixed initial and final quantum 3-geometries represented here by the spin-network states.
Applications

Einstein GR in $1+2$ space-time dimensions is a pure [BF] topological field theory. The spin foam approach yields a proper quantum gravity theory.

In real world ($1+3$ dimensions), GR is a BF-theory with constraint: not topological!

(simplicity: "the 2-form field $B$ should be a wedge product of cotetrad forms")

Initial approach (the categorical Barrett-Crane model) does not converge. How to find a spin foam formulation of the dynamics of LQG? This requires to find a suitable partition function obeying the constraints

$\Rightarrow$ a sum over certain functors in the categorical framework, (related to modular categories; Crane,Yetter).

Recent progress (EPRL model, for Engle-Pereira-Rovelli-Livine): a ["weak"] quantum implementation of the simplicity constraint was proposed. Crane interprets it through a time functor

$$F_\gamma : \text{Repr}(\text{SO}(3)) \rightarrow \text{Repr}(\text{SL}(2, \mathbb{C})) : j \rightarrow (j, \gamma j).$$

Here the real number $\gamma \neq 0$ is the Barbero-Immirzi parameter.
Categories and quantum gravity

Many objects and structures involved in QGR approaches are categories and functors, with combinatorial character, and / or with deep analogies with cobordisms and QFTs (not surprising ?) The old (Barret-Crane) reference model was constructed in categorical framework (now with mathematical generalizations). Now categories are a recognized frame for present research.
Facts

- The structure of GR (in fact of any metric-like theory) is well express by [some sort of] cobordims;
- A causal structure (e.g. causet) is a category;
- (Classical or quantum) Field theories are functors from some geometrical category (∼ cobordisms) to an algebraic one (∼ Hilb). (both tensor †-categories!)
- Quantum Physics has a nice formulation in the topos context (geometry-agebra-logic)
- Most QGR models have (implicit or explicit) categorical expressions
Categorification

Categorification : sets $\rightarrow$ categories $\rightarrow$ 2-categories

A 2-category has also \textit{2-morphisms} (morphisms between its morphisms)
Categorification ” adds dimensions.” $\rightarrow$ ..... $n$-categories

A tensor category is in fact a 2-category with one object : $=$ \textit{monoidal category} : tensorification $\sim$

categorification $\sim$ quantization

\textbf{Categorification program} (Baez, Crane, Barrett...):
\textbf{Set} (with maps) $\sim$ \textbf{Cat} (with functors);
categories $\sim$ higher categories.
Presently applied to QGR
quantum gravity as a category of functors $A \rightarrow B$ ?

$A =$ a ” geometrical ” [discrete] category (space-time) (category theory provides many candidates)

$B =$ an ” algebraic” field category (a suitable subcategory of the unitary representations of the Lorentz algebra.)

*Barret-Crane model* was the first explicitly constructed in the categorical framework

Other models (issued from canonical quantization) may be reformulated (through spin-foams) in the context of categories.

A sum over states, interpreted as a *sum over certain functors* assigns quantum amplitudes to some elements of the configuration;

A summation over the different possible combinatorial configurations provides a *partition function* (= discrete version of the functional integral).
Conclusions

Present physics can be nicely expressed in

The categorical framework allows us to formulate present and tentative physics:

- very often in a natural and fundamental way
- relational
- permitting different points of view: geometric, algebraic, logical...
- allows natural generalizations (e.g., of geometry)
- → also, new epistemology and ontology for space-time, matter, ...

Well adapted for future physics (quantum gravity).
All [?] present attempts find a nice categorical expression.
Only a beginning
Other works

**Geometry** Categories allow many ways to generalize geometry → appear as a natural framework for quantum geometry.

- Topos Theory and Spacetime Structure JERZY KROL’ ’ At sufficiently small distances, those of order of the Planck length or bigger, the logic assigned to the description of spacetime regions, is weakened from classical to intuitionistic logic of some topoi.
- Synthetic differential geometry (Lawere): the ”real number object ” $\mathbb{R}$ in the topos (generalization of $\mathbb{IR}$ in $\textbf{Set}$) contains small and large infinites...
- Grothendieck topoi generalizes topology
- Sheafs and co-sheaves, presheaves, fiber bundles are in fact topoi...
- exotic smoothness structure, non commutative geometry...
- Diffeology : a categorical generalization of differential manifolds, very adapted to symplectic geometry. (Souriau, Iglesias...)
Modular theory

Given a Von Neumann algebra (operators on an Hilbert space) the modular theory assigns an internal thermal time flow to each state. This may be a solution for the problem of time in grq. There is a categorical generalization.

Qunatum theory

(JERZY KROL)

Given a Hilbert space $H$ of states and the lattice of all projections on closed subsets we can always choose the maximal Boolean algebras $BH$ of projections. Next, given any complete Boolean algebra we have Boolean- valued model of ZFC, $Sh5BH = V^{BH}$ which is known to be the topos of sheaves of sets on $BH$. Theorem (Takeuti). All real numbers from the object of real numbers $R$ in $Sh(BH)$, are exactly in one to one correspondence with the self-adjoint linear operators in $H$ which are in $BH$. 
References


John C. Baez *Higher-Dimensional Algebra and Planck-Scale Physics* arXiv:gr-qc/9902017v1


